

Small Publications in Historical Geophysics

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**An Analysis of the First Gravimetric Investigations of the
Earth's Flattening and Interior using Clairaut's Theorem**

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1. Background

The flattening of the Earth, nowadays determined with high precision from perturbations of satellite orbits, was earlier determined by arc measurements or gravity measurements at different latitudes on the Earth's surface. A historical review of the earliest studies of the Earth's flattening may be found in Todhunter (1873); see also Levallois (1988).

The arc measurements, combining astronomical latitude observations with triangulation along meridians, constituted the main method. They started with the famous expeditions sent out in 1735 and 1736 by the French Academy of Sciences, one to Peru (now Ecuador) close to the equator, the other to Sweden (now Sweden and Finland) close to the arctic circle. In addition to these there was an arc measurement in France itself, and a little later one was performed in South Africa and one in the Papal States (today's Italy). Thus, by the middle of the century there were 5 arc measurements at hand. They showed beyond doubt that the Earth was flattened at the poles, but the amount of the flattening turned out to be difficult to determine; the results were inconsistent.

Most of the arc measurement expeditions were charged with the task of making gravity measurements as well. There were also separate gravity expeditions carried out. The gravity measurements, using pendulums, were less time-consuming than the arc measurements and could, therefore, more easily be carried out in several different latitudes. Thus, by the time there were 5 arc measurements completed there were no less than 19 gravity measurements performed in the world, although some of them quite close to each other. The gravity measurements clearly confirmed that the Earth was flattened at the poles. However, at this time there were no statistical methods to analyse a large amount of data, i.e. to deal with overdeterminations. Probably due to this fact no conclusive result on the value of the Earth's flattening from these gravity data was obtained.

Our intention here is to use regression analysis for the above-mentioned gravimetric data, and then to apply Clairaut's theorem to determine the corresponding flattening of the Earth, including its uncertainty. This result will be compared with those from the contemporary arc measurements, and we will see whether it would allow any conclusion also on the internal structure of the Earth. Finally, the result as well as the gravimetric data themselves will be compared with modern results and gravity values.

2. Clairaut's flattening theorem and the Earth's interior

The possibility to determine the figure of the rotationally deformed Earth from gravity measurements was discovered by Clairaut (1743). Assuming the shape of the Earth to be that of an equipotential surface of the Earth's gravity field, he derived the formula nowadays known as Clairaut's theorem, allowing the flattening of the Earth ellipsoid to be calculated from gravity data.

The flattening is defined by

$$f = (a - b)/a \quad (1)$$

where a is the semi-major axis and b the semi-minor axis of the ellipsoid. Correspondingly we have the gravimetric flattening

$$f^* = (\gamma_b - \gamma_a)/\gamma_a \quad (2)$$

where γ_a is the (normal) gravity at the equator and γ_b the (normal) gravity at the poles. In addition we need the ratio between the centrifugal force and the gravity at the equator,

$$q = \omega^2 a / \gamma_a \quad (3)$$

ω being the Earth's rotational velocity. Clairaut's elegant theorem now reads:

$$f + f^* = 5q/2 \quad (4)$$

Its original appearance is reproduced in Figure 1. Moreover, Clairaut proved the expression for gravity γ as a function of latitude φ :

$$\gamma = \gamma_a(1 + f^* \sin^2\varphi) \quad (5)$$

If we have a sufficient number of gravity measurements spread over different latitudes, a regression curve according to (5) may be fitted to the data. This will yield an estimate of the gravimetric flattening f^* . Inserting this into Clairaut's theorem (4) we obtain an estimate of the geometric flattening f . This requires the number q to be accurately known, which it was already at that time.

The formulae (4) and (5) are not quite exact; to their right-hand sides can be added small higher order terms in a series expansion (Helmert, 1884; Heiskanen & Moritz, 1967). As already the next term would contribute only some parts in a thousand, an insignificant amount in our case, we will keep to the original formulae above.

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F I G U R E

précédente par Π , nous pourrons nous contenter de la diviser par $2cA$, qui exprime la pesanteur sur le Globe composé de la même manière que le Sphéroïde. Nous aurons donc

$$2\delta + \varphi - \frac{3D}{5A} = \frac{P - \Pi}{\Pi}.$$

Si on se rappelle présentement qu'on a prouvé dans le §. XXXVII. qu'afin que le Fluide placé sur la surface du Sphéroïde pût être en équilibre, il falloit que $10\delta A - 2D = 5A\varphi$, on verra qu'on peut mettre dans la quantité précédente $3\delta - \frac{1}{2}\varphi$ à la place de $\frac{3D}{5A}$.

La substitution faite, il viendra

$$\frac{5}{2}\varphi - \delta = \frac{P - \Pi}{\Pi}. \text{ Or } \frac{5}{4}\varphi \text{ est } \delta. \text{ XXX.}$$

la valeur de ε ou de l'ellipticité du Sphéroïde homogène : donc $\frac{P - \Pi}{\Pi} = 2\varepsilon - \delta$.

Figure 1. A page from Clairaut's book (1743) where his theorem first appears, third line from the bottom. (His φ corresponds to our q , his δ to our f , and his $(P - \Pi)/\Pi$ to our f^c .)

Although Clairaut's theorem is independent of the internal structure of the Earth, the flattening is not. Clairaut himself (1743) showed that the flattening is dependent on the density distribution inside the Earth, and bound by two extreme values. A completely homogeneous Earth would yield the maximum flattening,

$$f_h = f_h^* = 5q/4 \quad (6)$$

The more the density gradient towards the centre of the Earth increases, the more the flattening decreases. The limiting case of an Earth with all its mass concentrated in the centre would yield the minimum flattening,

$$f_c = f_c^*/4 = q/2 \quad (7)$$

Applying $q = 1/288$ Clairaut finds

$$1/230 > f > 1/576$$

Hence, determinations of the Earth's flattening could, if accurate enough, also give insight into the internal density distribution of the Earth.

We may remark here that in the somewhat artificial situation of an inverse density distribution, the flattening would be larger than for a homogeneous Earth. In the limiting case of all mass concentrated to a surface shell one would have $f_s = 5q/2$, $f_s^* = 0$ (Kopal, 1960). Thus, for inverse density distributions $1/115 > f > 1/230$.

3. The gravimetric data

The gravimetric data used in our analysis are taken from a compilation by Mallet (1772), who carefully reviewed all gravity measurements made in the world until the end of the 1750s. All measurements made after 1730 have been used; they benefit from the development of accurate pendulums in London and Paris. The gravity measurement closest to the pole is the one made near the arctic circle in connection with the arc measurement in Sweden/Finland (Maupertuis, 1738). The gravity measurements closest to the equator are those made in connection with the arc measurement in Peru/Ecuador (Bouguer, 1749). In total there are 19 gravity measurements (close to sea level); all the data are collected in Table 1.

The original pendulum data are given as lengths, expressed in French lines, of pendulums with an oscillation half-period of one second. The lengths

Table 1. Early gravity measurements of the world (φ = latitude in degrees and minutes, l = pendulum length in lines, g = gravity in gals).

Station	Year	Observer	φ	l	g
Pello (Sweden/Finland)	1736	Maupertuis	66 48	441.14	982.16
Uppsala (Sweden)	1741	Celsius	59 51	440.86	981.54
St. Petersburg (Russia)	1757	Grischow	59 56	441.00	981.85
Reval (Russia/Estonia)	1757	Grischow	59 26	440.93	981.69
Pernau (Russia/Estonia)	1757	Grischow	58 26	440.92	981.67
Dorpat (Russia/Estonia)	1757	Grischow	58 26	440.91	981.65
Arensburg (Russia/Estonia)	1757	Grischow	58 16	440.88	981.58
Leiden (Netherlands)	1755	Lulofs	52 09	440.71	981.20
London (Great Britain)	1731 -	Graham a. o.	51 31	440.60	980.96
Paris (France)	1735 -	Mairan a. o.	48 50	440.57	980.89
Rome (Papal States/Italy)	1745	Jacquier	41 54	440.28	980.25
Kingston (Jamaica)	1732	Campbell	18 00	439.40	978.29
Guarica (Haiti)	1743	Bouguer group	19 46	439.32	978.11
Petit Goave (Haiti)	1735	Bouguer group	18 27	439.31	978.09
Portobelo (Panama)	1735	Bouguer group	9 34	439.12	977.66
Panama (Panama)	1735	Bouguer group	8 55	439.20	977.84
Puntapalmar (Peru/Ecuador)	1740	Bouguer group	0 02	438.96	977.31
Jama (Peru/Ecuador)	1740	Bouguer group	- 0 09	439.00	977.40
Cape Town (South Africa)	1751	Lacaille	- 33 55	440.07	979.78

Table 2. Latitudes and gravity values used in the analysis; figures based on Table 1 as described in the text.

φ	g
66 48	982.16
59 51	981.54
58 54	981.69
52 09	981.20
51 31	980.96
48 50	980.89
41 54	980.25
33 35	979.78
19 06	978.10
18 00	978.29
9 14	977.75
0 06	977.36

have been transformed into cm through the relation 1 line = 0.225583 cm (Levallois, 1988). Then the gravity values, expressed in gals (cm/s^2), have been calculated from

$$g = \pi^2 l / t^2$$

l being the pendulum length and t the half-period. Both the original pendulum data and the calculated gravity values are shown in Table 1.

As can be seen from Table 1, one observer has sometimes made several measurements at approximately the same latitude. Such measurements should be merged into one before going into a regression analysis. Accordingly, the five Russian-Baltic measurements have been averaged into one, and the six South American measurements by the Bouguer group into three. This leaves us with 12 independent gravity values at different latitudes to be put into our analysis; they are listed in Table 2.

It is worth noticing that at least one of the original pendulum clocks or gravimeters is preserved and still functioning. This is Celsius' instrument, constructed for him by Graham in London, which has been in operation at the Astronomical Observatory of Uppsala since Celsius brought it there.

4. Applications of Clairaut's theorem

We start our analysis by making a regression of the data of Table 2 to fit a curve of type (5) to the data. This means determining γ_a and f^* in (5), including their standard errors, through a least squares adjustment of the g values. The outcome of this operation is

$$\gamma_a = 977.65 \pm 0.11 \text{ gal}$$

$$f^* = 1/(177.6 \pm 6.8)$$

The data and the regression curve are shown in Figure 2. It is tempting already here to make comparisons with modern values but we postpone that till the end of the publication; at this stage we concentrate on analyzing the historical data as they are, without knowing anything of future measurements.

The standard error of a gravity value (in this case at the equator) is as small as 0.1 gal. The spread of gravity values around the regression curve seems quite reasonable, although there are two deviations (with opposite signs) of 0.3 gal. Thus there should be no reason to exclude any data because of

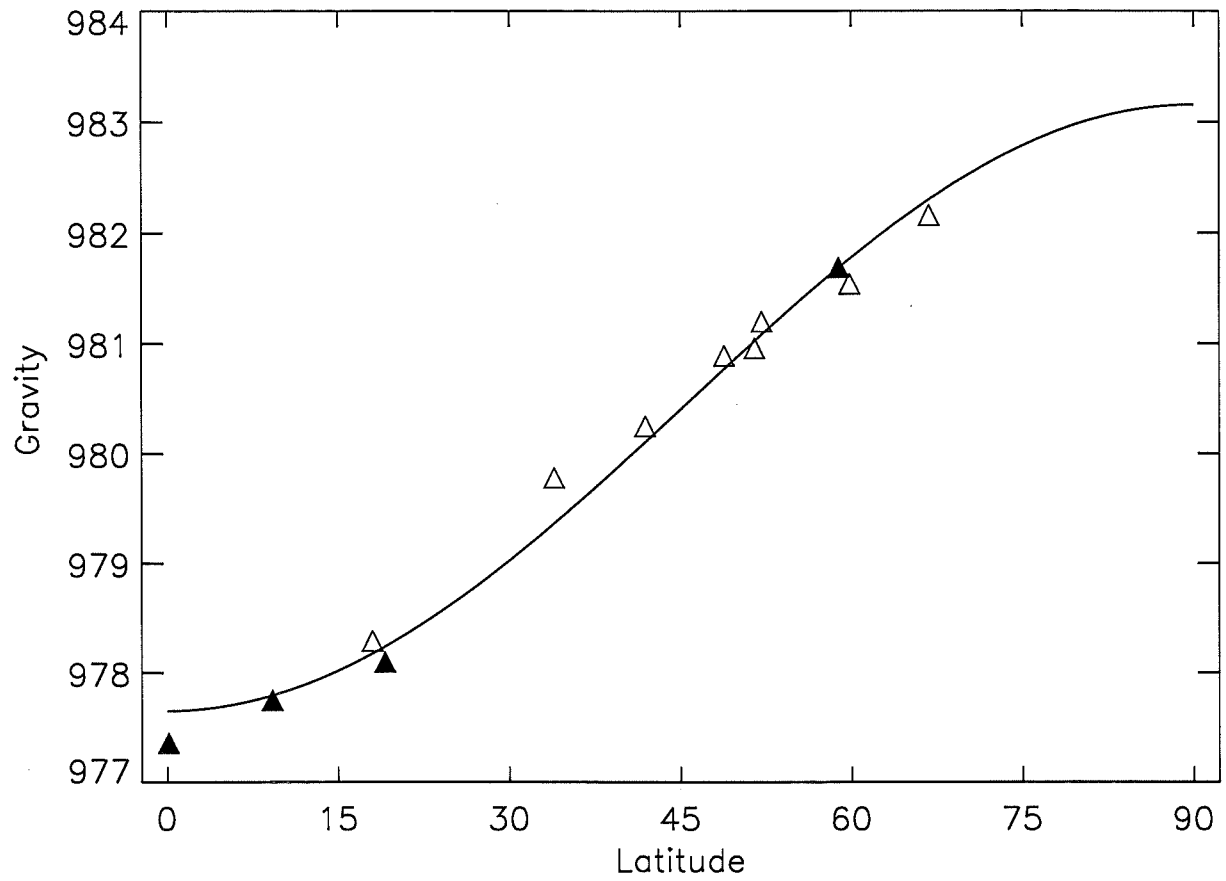


Figure 2. Regression curve fitted to the gravity data of Table 2.

obvious non-random errors. Consequently we also accept the obtained estimate of the gravimetric flattening for further processing.

Before putting f^* into (4) we have to fix a value of the equatorial ratio q . Bouguer (1749) gives $q = 1/288.5$. This agrees well with the value we can determine ourselves from (3). In this expression ω is found from the known length of a sidereal day, 86 164 s, yielding $\omega = 7.2921 \cdot 10^{-5} \text{ s}^{-1}$, a has to be taken from the arc measurements, indicating $a \approx 6\,380$ or $6\,390$ km, and γ_a is already determined above. The result is

$$q = 1/(288.0 \pm 0.2)$$

where the standard error has been somewhat loosely estimated from the uncertainty in a . Obviously q can be considered error-free in comparison with f^* .

Now we are in the position to insert f^* and q into Clairaut's theorem (4). We obtain

$$f = 1/(327.9 \pm 12.6)$$

This is a very good estimate of the Earth's flattening, the standard error being surprisingly small.

Furthermore, this estimate allows us to draw conclusions on the Earth's interior. Applying a t distribution with 10 degrees of freedom (12 - 2 overdeterminations) we find a 95 % confidence interval for f of

$$1/299.8 > f > 1/356.0$$

Thus we have $1/230 \gg f$, showing that the Earth cannot be a homogeneous body; the density has to increase towards the centre of the Earth. Clairaut (1743) already notes that the gravity data he has at hand points in such a direction.

5. Comparisons with contemporary arc measurements

As mentioned in the Background chapter, the flattening investigations considered as the main ones were the 5 great arc measurements performed. In order of latitude they were:

1. Peru/Ecuador (equator) 1735 - 1744 under the leadership of Bouguer.
2. South Africa 1750 - 1753 under the leadership of Lacaille.

3. Papal States/Italy 1750 - 1755 under the leadership of Boscovich.
4. France 1739 - 1744 under the leadership of Cassini de Thury.
5. Sweden/Sweden-Finland (arctic circle) 1736 - 1737 under the leadership of Maupertuis.

However, it was difficult to find a common flattening from these arc measurements. Calculating the flattening from pairwise measurements with sufficient latitude differences, i.e. from all pairs containing either the equatorial or the arctic measurement (or both), resulted in a wide spread of values (see e.g. Mallet, 1772),

$$1/144 > f > 1/352$$

The equatorial and the arctic measurement in combination yielded $f = 1/215$. (These computations by Mallet are correct, in contrast to his computations from gravity data, which apparently rest on an assumption valid only for a homogeneous Earth.) On the whole, several authors were led to assume a flattening of $f \approx 1/230$, including Lalande (1764) in his authoritative text-book in astronomy.

Comparing our gravimetric determination of the flattening with that possible to make from the arc measurements, the gravimetric one is obviously superior. The accuracy is considerably higher and the result rests on 10 overdeterminations instead of 3. In particular, while the gravity measurements allow us to deduce the inhomogeneity of the Earth, the arc measurements do not allow any conclusion at all in that respect. (If anything, they rather indicate a homogeneous Earth.)

6. Comparisons with modern results

It is now time to compare the obtained results with modern ones, based on the GRS 1980 ellipsoid. For our three main quantities we have, according to Moritz (1980),

$$\gamma_a = 978.03 \text{ gal}$$

$$f^* = 1/188.6$$

$$f = 1/298.3$$

The obtained flattenings f^* and f in chapter 4 can be seen to deviate by about twice their standard errors from the modern values. The upper limit of the 95 % confidence interval for f , $1/299.8$, comes very close to the modern value

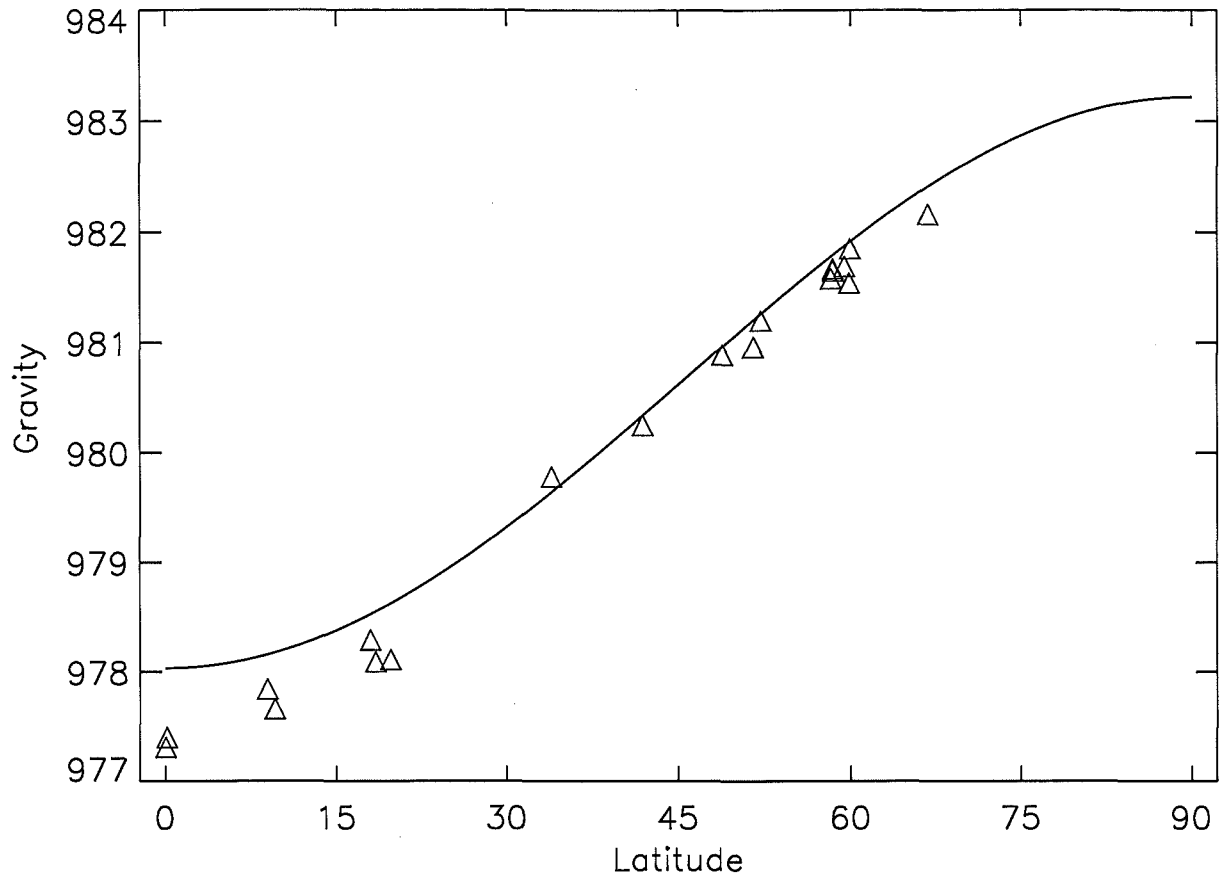


Figure 3. Normal gravity curve according to GRS 1980 compared with the gravity observations of Table 1.

above. Thus the result must be said to be quite good also from today's point of view.

The obtained equatorial value of the gravity, on the other hand, deviates by between three and four times its standard error. This calls for further comparisons of the gravity data themselves. In Figure 3 we show all the 19 gravity measurements in relation to the normal gravity curve of the GRS 1980 ellipsoid, which is the modern version of the curve of Figure 2. All measurements are made close to sea level so that deviations from normal gravity should be more or less negligible. Figure 3 reveals what seems to be two systematic errors: a small general one, and one connected to the Bouguer equatorial group.

The general systematic error amounts to -0.2 gal. The reason behind this is most probably an effect discussed already by Bouguer (1749): The pendulum measurements are not performed in vacuum. Bouguer estimated this effect at about -0.10 lines in pendulum length, which corresponds to about -0.20 gal, agreeing nicely with the error seen in Figure 3. Correcting for this effect, we still find a systematic error, in the measurements of the Bouguer group, amounting to -0.4 gal. Its origin remains unknown (although a temperature effect might be suspected); it is too large to be caused by local gravity anomalies. This error has a major influence on the flattening result because of its closeness to the equator, i.e. to one end of the regression curve.

7. Concluding remarks

Our analysis of the gravity measurements in the world made around the 1740s has shown that they allow a much better determination of the flattening of the Earth than the famous contemporary arc measurements. We obtain $f = 1/(328 \pm 13)$. Moreover, according to Clairaut's theory this result demonstrates that the Earth is an inhomogeneous body, with the density increasing towards its centre. In reality this was not demonstrated until nearly half a century later, when Cavendish, through determining the gravitational constant, could show that the Earth's mean density was larger than its surface one. At the same time also Laplace (1799), with more gravity data at hand, reached a flattening close to the one above.

The discrepancy between the obtained flattening and the real one ($f = 1/298$) almost falls within the 95 % confidence interval. Most of the discrepancy can be attributed to a systematic error of unknown origin detected in the equatorial gravity measurements of the Bouguer group.

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