On the First Conformal Projection in Official Topographic Mapping

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1.   Introduction

As is well known, the curved surface of the Earth is not possible to project onto a plane, i.e. a map, without distortions. One way of handling this problem is to make an equal-area projection, whereby the sizes of objects are preserved in the map. That will, however, make angles distorted and, thereby, the shape of the objects. Another way is to make an equal-angle or conformal projection, whereby angles are preserved and, thereby, the shape of objects in the map. That will, instead, make the sizes distorted. Conformal projections are the ones preferred for the fundamental accurate maps of a country, like topographic maps, since preserving shapes and angles are much more important in this case than preserving sizes.

The mathematical theory of conformal projections was originally developed by Lambert (1772). However, it was not adopted and applied by official mapping authorities until about 1920. A wider mathematical theory of conformal projections was developed by Gauss (1825, 1844). Neither this was adopted and applied for official maps until about 1920. So, the use of conformal projections in official mapping did not spread over the world until during the 1900s. (We here disregard the special case of Mercator's projection for nautical charts.) In the standard work on map projections by Snyder (1987) it is pointed out that the Lambert conformal projection was overlooked until it was introduced in the United States of America in the early 1900s.

In this perspective it is interesting to note that a conformal map projection was actually introduced in Scandinavia already 100 years earlier. In 1817 Sweden and Norway, then in a union, agreed on a conformal projection for their planned topographic map series. The theory behind it was published by Spens (1817), without knowing of Lambert's work. It was adopted and applied immediately in Sweden. It is actually a kind of Lambert projection, although a different version of it, a "Scandinavian" Lambert projection. Some decades later Denmark went a related way.

We will here study the pioneering conformal map projection of Scandinavia from two different aspects. First, what are the similarities and differences between the Scandinavian Lambert projection, or the Spens projection, and the original Lambert projection as later applied in other countries? Second, why was the projection introduced half a century after it was originally discovered but a whole century before other countries did something similar?

2. Comparison of the projection methods of Lambert and Spens

The theory of conformal map projections introduced by Lambert (1772) and Spens (1817) makes use of conic projection surfaces, the cone being tangent to or intersecting the Earth ellipsoid along some parallel(s). The basis for both Lambert's and Spens' derivation of the fundamental formula for the projection is the fact that for a conformal projection the scale distortion is the same in all directions. The scale distortion along a meridian can be written

$$h = \frac{-dm}{M(\varphi)\, d\varphi} \tag{1}$$

In the denominator here $d\varphi$ is a small change in latitude φ causing a displacement along the meridian on the ellipsoid of $M(\varphi)\, d\varphi$, where $M(\varphi)$ is the meridional radius of curvature. In the numerator dm is the corresponding small displacement on the map, m being the map distance from the pole. The scale distortion along a parallel can, in a similar manner, be written

$$k = \frac{nm}{N(\varphi)\, \cos\varphi} \tag{2}$$

where $N(\varphi)$ is the perpendicular radius of curvature and n is a constant depending on the cone's angle. The condition for a conformal projection is

$$h = k \tag{3}$$

Inserting (1) and (2) into (3) yields a differential equation. When solved this produces the conformal projection formula

$$m = C\left[\tan(45° - \frac{\varphi}{2}) \left[\frac{1 - e\sin\varphi}{1 + e\sin\varphi} \right]^{e/2} \right]^{n} \tag{4}$$

The projection formula gives the distance m on the map from the pole to a point as a function of the latitude φ of the point; e is the eccentricity of the ellipsoid, and C is an integration constant in which is embedded the semi-major axis a of the ellipsoid.

Lambert and Spens both arrive at (4) following the above principles. Details in their procedures differ, but these differences are not too important. Once formula (4) has been established, however, there arises the question of

how to determine the two constants appearing there, C and n. And here Lambert and Spens go different ways.

Lambert (1772) does not discuss very much the constants in the projection. He notes that one can choose a cone tangent to the ellipsoid creating an error-free (equidistant) standard parallel, which will result in certain values of the constants. He also notes that one can choose a cone intersecting the ellipsoid creating two such standard parallels, which will then result in other values of the constants. (These parallels are not identical to parallels of intersection with the cone.) Between the standard parallels there will be a latitude-dependent scale distortion making everything somewhat diminished, outside the standard parallels there will be a latitude-dependent scale distortion making everything enlarged.

Spens (1817), on the other hand, develops a method of finding values of the projection constants that will minimize the scale distortions over the area to be mapped. Instead of choosing beforehand two standard parallels, he defines a northern and a southern limiting latitude for the mapping area and then puts up the following conditions for optimizing the distribution of the projection errors within the area: First, the scale distortion has, somewhere in the middle, a minimum value, $h_0 = k_0 < 1$. This minimum occurs at the latitude where the derivative of k (or h) is equal to zero,

$$\frac{dk}{d\varphi} = 0 \qquad (5)$$

where k is given by (2). Second, the scale distortion at the northern and the southern limiting parallels, $h_N = k_N = h_S = k_S > 1$, should be equal to the inverted value of the minimum,

$$k_N = k_S = \frac{1}{k_0} \qquad (6)$$

In this way Spens does not need to choose (guess) suitable standard parallels as Lambert does, but can use the limiting parallels of the map to find those constants in (4) that will optimize the distribution of the projection errors.

3.   Comparison of the first official applications

When Spens (1817) published the mathematical treatment of his conformal projection he also dealt with its application (cf. Rosén, 1876). He designed a projection to be used for the planned topographic map series of

Sweden and Norway, at that time, as mentioned, in a newly formed union. For this purpose Spens, according to above, was to define the two limiting parallels of the area to be mapped. As this area was Sweden and Norway, together forming the Scandinavian peninsula, the southern parallel should be close to the Swedish south coast in the Baltic Sea and the northern parallel close to the Norwegian north coast at the Arctic Ocean. However, Spens considered that a projection error on a map of the wilderness in the northernmost parts of Sweden and Norway was less important. Therefore, he put the northern limiting parallel at the northernmost part of the Gulf of Bothnia instead. The latitudes of these limiting parallels were fixed at

$$\varphi_S = 55°21'19.4''$$
$$\varphi_N = 65°50'20.4''$$

Within these parallels the projection errors, i.e. the scale distortions, were to be minimized. Using (5) and (6) together with (4) this resulted in the following distribution of the scale distortion; see Spens (1817) and Arosenius (1859):

$k_S = 1.0021$	$\varphi_S = 55°21'19.4''$
$k_s = 1.0000$	$\varphi_s = 56°57'40.0''$
$k_0 = 0.9979$	$\varphi_0 = 60°44'29.6''$
$k_n = 1.0000$	$\varphi_n = 64°22'48.0''$
$k_N = 1.0021$	$\varphi_N = 65°50'20.4''$

The error-free parallels here, characterized by $k_s = k_n = 1.0000$, would correspond to the optimum choice of standard parallels, but with this method, as stated earlier, there is no need of choosing standard parallels – they are just a result of the optimization process.

The original Lambert projection was, as mentioned in the Introduction, not officially applied until the early 1900s in the United States of America. Inspiration came from a military application of it in France during the First World War, taken up and discussed by Deetz (1918, 1918a). Based on this, Adams (1918, 1918a) presented a complete application of the Lambert projection for use in the United States. Deetz and Adams use the "ordinary" way of determining the constants of the projection: They choose two standard parallels to be error-free, from which then the projection constants are found. For choosing the standard parallels they recommend a general thumb rule: The standard parallels are chosen at 1/6 and 5/6 of the north-south distance across the area to be mapped. This Lambert projection was later adopted and

applied by the majority of the states for the official State Plane Coordinate System.

Now, how does the thumb rule of Deetz (1918) and Adams (1918a) compare with the optimization method of Spens (1817)? According to the thumb rule the standard parallels of Spens should be located at 1/6 of the total latitude difference from, respectively, the southern and the northern limiting parallels. Using the latitude data above we find the following actual fractions:

Spens, southern   1/6.5
Spens, northern   1/7.2

Thumb rule   1/6

This shows that, although the fraction according to the thumb rule is not too bad, the fraction is latitude-dependent and that the fractions for the southern and northern standard parallels are not equal. (Besides, Deetz and Adams put their parallel of minimum scale distortion in the middle between the standard parallels. The latitude data of Spens above show that this is not quite the case: The minimum scale distortion latitude there is somewhat more than 4′ to the north of the average latitude of the standard parallels.) On the whole, the early optimization method of Spens yields a somewhat more favourable distribution of the scale distortion than the later thumb rules do.

4.   Mathematical discovery versus official introduction

The time from Lambert (1772) to Deetz (1918) and Adams (1918) is more than 140 years. This is thus the time it took from the mathematical discovery to the official introduction of the original Lambert conformal projection. As mentioned in the Introduction a similar waiting time relates to the Gauss conformal projection. The time from Gauss (1825) to Krüger (1919) is nearly 100 years. For both the two fundamental theories of conformal projections the time from discovery to official introduction is extremely long. On the other hand, the corresponding waiting time for the Spens conformal projection is 0 years! These extremely different waiting times deserve some comments.

Lambert as well as Gauss were primarily mathematicians and wrote their papers mainly from a mathematical point of view. They did not bother too much about presenting how to calculate projection constants for a certain application. Neither did they calculate any projection tables necessary for practical use when constructing detailed maps of a country. Thus for a decision-maker in an official mapping authority in the 1800s it was probably

not easy to adopt a conformal projection, neither according to Lambert nor Gauss, for the fundamental mapping.

Spens, on the other hand, was a geodesist at the official mapping authority of Sweden. When he on his own, lacking knowledge of Lambert's work, invented the conformal projection, he both developed the mathematical theory, determined suitable projection constants for the mapping purpose, and calculated tables for the practical construction of the topographic maps. He so to speak did the whole work, from the theoretical beginning to the practical end. This probably made it easier for the mapping authority to decide on this new kind of projection, although such projections were not used anywhere else in the world. An example of the projection tables of Spens (1817) is shown in Figure 1.

The original Lambert projection was introduced officially first after Adams (1918) had calculated useful projection tables for mapping, Adams being a state geodesist in the United States. And the Gauss projection (Transverse Mercator projection) got its break-through first after Krüger (1912, 1919) had published practically useful formulae and numerical examples, Krüger being a state geodesist in Germany (Prussia). In the latter case the problem had been complicated by Gauss' way of working. Gauss applied his own projection in his calculations of the triangulation of Hannover but he only published his fundamental mathematical ideas. Hence nobody knew how it all worked. First after Schreiber (1866, 1897) had further figured things out there was enough knowledge to allow Krüger to make the projection useful for Germany as well as the outer world. In contrast to this, Spens one hundred years earlier made everything himself in one and the same paper. That seems to be the main reason why Scandinavia introduced a conformal projection into its official topographic maps long before others did so.

5.   What happened later on?

When constructing a map based on Lambert's/Spens' projection there is one more parameter that needs to be specified: the central meridian. This is the meridian where north is exactly upwards on the map. The other meridians will not be parallel to this one; they will converge towards the pole, the more they are distant from the central meridian. When Spens (1817) constructed his projection for a topographic map series of Sweden and Norway he fixed the central meridian at a longitude 5° west of the Stockholm observatory, reasonably close to the Swedish-Norwegian border.

After the Swedish-Norwegian agreement in 1817 to adopt Spens' projection for a topographic map series of the two union countries, the Swedes

Grader.	Minuter.	Parallelers Radier i Sv. Fot.	Diff. 1'	Corrections-Factorers Logarithmer	Diff.	Project. Scala då Hutv. Scal. =1: 20000.
62	0	11 565 944		0. 000 8052		1: 20037. 1
	5	11 534 728	6 243	0. 000 7906	— 146	1: 20036. 4
	10	11 503 511	6 243	0. 000 7750	— 156	1: 20035. 7
	15	11 472 293	6 244	0. 000 7584	— 166	1: 20035. 0
	20	11 441 073	6 244	0. 000 7409	— 175	1: 20034. 1
	25	11 409 852	6 244	0. 000 7224	— 185	1: 20033. 3
	30	11 378 629	6 245	0. 000 7030	— 194	1: 20032. 4
	35	11 347 403	6 245	0. 000 6826	— 204	1: 20031. 5
	40	11 316 175	6 246	0. 000 6612	— 214	1: 20030. 5
	45	11 284 945	6 246	0. 000 6388	— 224	1: 20029. 4
	50	11 253 713	6 246	0. 000 6155	— 233	1: 20028. 4
	55	11 222 480	6 247	0. 000 5911	— 244	1: 20027. 2
63	0	11 191 245	6 247	0. 000 5658	— 253	1: 20026. 1
	5	11 160 007	6 248	0. 000 5395	— 263	1: 20024. 9
	10	11 128 767	6 248	0. 000 5122	— 273	1: 20023. 6
	15	11 097 524	6 249	0. 000 4839	— 283	1: 20022. 3
	20	11 066 279	6 249	0. 000 4546	— 293	1: 20020. 9
	25	11 035 032	6 249	0. 000 4243	— 303	1: 20019. 5
	30	11 003 782	6 250	0. 000 3930	— 313	1: 20018. 1
	35	10 972 529	6 251	0. 000 3607	— 323	1: 20016. 6
	40	10 941 273	6 251	0. 000 3274	— 333	1: 20015. 1
	45	10 910 015	6 252	0. 000 2930	— 344	1: 20013. 5
	50	10 878 754	6 252	0. 000 2576	— 354	1: 20011. 9
	55	10 847 491	6 253	0. 000 2213	— 363	1: 20010. 2
64	0	10 816 224	6 253	0. 000 1839	— 374	1: 20008. 5
	5	10 784 954	6 254	0. 000 1454	— 385	1: 20006. 7
	10	10 753 680	6 255	0. 000 1059	— 395	1: 20004. 9
	15	10 722 403	6 255	0. 000 0654	— 405	1: 20003. 0
	20	10 691 123	6 256	0. 000 0239	— 415	1: 20001. 1
	25	10 659 840	6 257	9. 999 9813	— 426	1: 19999. 1
	30	10 628 554	6 257	9. 999 9376	— 437	1: 19997. 1
	35	10 597 264	6 258	9. 999 8929	— 447	1: 19995. 1
	40	10 565 970	6 259	9. 999 8471	— 458	1: 19993. 0
	45	10 534 672	6 260	9. 999 8002	— 469	1: 19990. 8
	50	10 503 371	6 260	9. 999 7523	— 479	1: 19988. 6
	55	10 472 066	6 261	9. 999 7034	— 489	1: 19986. 3
65	0	10 440 757	6 262	9. 999 6534	— 500	1: 19984. 0
	5	10 409 444	6 263	9. 999 6022	— 512	1: 19981. 7
	10	10 378 127	6 263	9. 999 5500	— 522	1: 19979. 3
	15	10 346 806	6 264	9. 999 4967	— 533	1: 19976. 8
	20	10 315 481	6 265	9. 999 4423	— 544	1: 19974. 3
	25	10 284 151	6 266	9. 999 3868	— 555	1: 19971. 8
	30	10 252 817	6 267	9. 999 3303	— 565	1: 19969. 2
	35	10 221 478	6 268	9. 999 2726	— 577	1: 19966. 5
	40	10 190 135	6 269	9. 999 2138	— 588	1: 19963. 8
	45	10 158 788	6 269	9. 999 1539	— 599	1: 19961. 1
	50	10 127 436	6 270	9. 999 0930	— 609	1: 19958. 3
	55	10 096 079	6 271	9. 999 0309	— 621	1: 19955. 4
66	0	10 064 717	6 272	9. 998 9676	— 633	1: 19952. 5

Figure 1. Conformal map projection tables of Spens (1817). Column to the far left: Latitude in degrees and minutes. Next column: Map distance from the pole in Swedish feet. Column to the far right: Scale distortion based on a standard scale of 1 : 20 000. Note the error-free parallel at latitude 64°22′48″ (cf. page 6).

immediately started working on carrying this into effect. Based on first order triangulation performed for the purpose the first maps appeared in 1826 (Ottoson & Sandberg, 2001), to begin with as military secrets. The Norwegians, however, did not have the resources to start up their work, and when they did so a decade later they decided to go their own way. They never implemented the Spens projection they had agreed to earlier but adopted a more common one instead (Arosenius, 1859; Seue, 1878). One might assume that the Norwegians did not feel happy about introducing a new kind of map projection that was not to be found anywhere else in the world.

Thus only the Swedes actually implemented the conformal projection designed for the whole Swedish-Norwegian union. (For northern Sweden the projection was later somewhat changed.) This had a peculiar long-term consequence. As the central meridian was close to the Swedish-Norwegian border most of Sweden got a graticule of meridians and parallels on the maps that was systematically tilted to the west of the northern direction. This topographic map series was the official one partly up to 1979, when the last of its map sheets was replaced by the present maps in the Gauss projection. Thus Sweden alone for more than 170 years had maps with a graticule designed for a Swedish-Norwegian union as a whole. But it was, remarkably enough, the first conformal topographic map series in the world!

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